MTH 508/608: Introduction to Differentiable Manifolds and Lie Groups
Semester 1, 2024-25

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This Lesson Plan is based on the topics covered in [1, 2].

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1 Differentiable manifolds

1.1 Review of multivariable differential calculus

1.1.1 Real-valued differentiable functions

(i) **Definition.** Let $f : U(\subset \mathbb{R}^n) \to \mathbb{R}$, where U is an open set. Then for $1 \leq k \leq n$, the k^{th} *partial derivative* $\frac{\partial f}{\partial x_k}$ at $a = (a_1, \ldots, a_n) \in U$ is defined by:

$$\left(\frac{\partial f}{\partial x_k}\right)_a = \lim_{h \to 0} \frac{f(a_1, \ldots, a_k + h, \ldots, a_n) - f(a)}{h}.$$

(ii) **Definition.** A function $f : U(\subset \mathbb{R}^n) \to \mathbb{R}$ is said to be *continuously differentiable* on U (in symbols $f \in C^1(U)$) if for $1 \leq k \leq n$, $\left(\frac{\partial f}{\partial x_k}\right)$ is well-defined and continuous on U.

(iii) A function $f : U(\subset \mathbb{R}^n) \to \mathbb{R}$ is said to be *differentiable* at $a \in U$ if there exists constants $b_1, \ldots, b_n$ and a function $r(x, a)$ defined on a neighborhood $V \ni a$ in U satisfying the following conditions.

(a) $f(x) = f(a) + \sum_{i=1}^{n} b_i(x_i - a_i) + \|x - a\| r(x, a)$.

(b) $\lim_{x \to a} r(x, a) = 0$.

(iv) **Theorem.** Let $f : U(\subset \mathbb{R}^n) \to \mathbb{R}$, where U is an open set. If f is differentiable at $a \in U$, then f is continuous at a, and $\left(\frac{\partial f}{\partial x_k}\right)_a$ exists for $1 \leq k \leq n$ and $b_k = \left(\frac{\partial f}{\partial x_k}\right)_a$. Conversely, if $\left(\frac{\partial f}{\partial x_k}\right)$ for $1 \leq k \leq n$ exist for each y in some neighborhood $V \ni a$ and are continuous on V, then f is differentiable at a.

(v) **Definition.** Let $f : U(\subset \mathbb{R}^n) \to \mathbb{R}$, where U is an open set. Then:

(a) f is said to be *r-fold continuously differentiable* (in symbols $f \in C^r(U)$) if all of its r^{th} order partial derivtaives exists at each $a \in U$ and are continuous on U.

(b) f is said to be *smooth* (in symbols) $f \in C^\infty(U)$) if $f \in C^r(U)$ for each $r \geq 1$.

(vi) **Definition.** A differentiable C^r curve in $\mathbb{R}^n$ is a continuous map $f:(a,b)\to\mathbb{R}^n$ such that each component function $f_i:(a,b)\to\mathbb{R}$ for $1\le i\le n$ satisfies $f_i\in C^r(a,b)$.

(vii) **Proposition (Chain rule).** Let $f:(a,b)\to U(\subset\mathbb{R}^n)$ be a diffrentiable curve, and let $g:U\to\mathbb{R}$ be differentiable at $f(t_0)$ for some $to\in(a,b)$. Then $g\circ f$ is diffetentiable at t_0 and we have:

$$\frac{d}{dt}(g\circ f)_{t_0}=\sum_{i=1}^{n}\left(\frac{\partial g}{\partial x_i}\right)_{f(x_0)}\left(\frac{dx_i}{dt}\right)_{t_0}.$$

(viii) **Definition.** We say a domain $U\in\mathbb{R}^n$ is *star-shaped with respect to* $a\in U$, if for each $x\in U$, the line segment $\overline{ax}\subset U$.

(ix) **Theorem (Mean Value Theorem).** Let $f:U(\subset\mathbb{R}^n)\to\mathbb{R}$ be differentiable and let U be star-shaped with respect to $a\in U$. Then given $x\in U$, there exists $\theta\in(0,1)$ such that:

$$f(x)-f(a)=\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)_{a+\theta(x-1)}(x_i-a_i).$$

(x) **Corollary.** Let $f:U(\subset\mathbb{R}^n)\to\mathbb{R}$ be differentiable and let U be star-shaped with respect to $a\in U$. If for $1\le k\le n$, $\left|\frac{\partial f}{\partial x_i}\right|<k$ on U, then for any $x\in U$, we have:

$$|f(x)-f(a)|<k\sqrt{n}|x-a|.$$

(xi) **Corollary.** If $f\in C^r(U)$, then at each $a\in U$, the value of any k^{th} order mixed partial derivative is independent of the order of differentiation.

1.1.2 Differentiable functions $\mathbb{R}^n\to\mathbb{R}^m$

(i) **Definition.** Let $f:U(\subset\mathbb{R}^n)\to\mathbb{R}^m$, where U is open. Then:

(a) f is said to be *differentiable of class r* (in symbols $f\in C^r(U)$), if $f_i\in C^r(U)$, for $1\le i\le m$.

(b) f is said to be *smooth* (in symbols $f\in C^\infty(U)$) is $f_i\in C^\infty(U)$, for $1\le i\le m$.

(ii) If $f : U(\subset \mathbb{R}^n) \to \mathbb{R}^m$ is differentiable on U, then its *Jacobian matrix* defined by

$$Df := \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

exists at each $a \in A$.

(iii) **Proposition.** A mapping $f : U(\subset \mathbb{R}^n) \to \mathbb{R}^m$ is differentiable at $a \in U$ (resp. on U) if and only if there exists an $m \times n$ matrix A of constants (resp. functions on U) and an m-tuple $R(x,a) = (r_1(x,a),\ldots,r_n(x,a))$ of functions on U (resp. $U \times U$) such that $\|R(x,a)\| \to 0$ as $x \to a$ and for each $a \in U$, we have:

$$F(x) = F(a) + A(x-a) + |x-a|R(x,a).$$

If such $R(x,a)$ and A exists, then A is unique and $A = Df$.

(iv) **Theorem.** Let $f : U(\subset \mathbb{R}^n) \to \mathbb{R}^m$, where U is open, and let U be star-like with respect to $a \in U$. If f is differentiable on U with $\left|\frac{\partial f_i}{\partial x_j}\right| \le k$ for $1 \le i \le m$ and $1 \le j \le n$, for every $a \in U$. Then:

$$|F(x) - F(a)| \le \sqrt{nm}k|x-a|.$$

(v) **Theorem(Chain Rule).** Let $f : U(\subset \mathbb{R}^n) \to V(\subset \mathbb{R}^m)$ and let $g : V \to \mathbb{R}^p$. If f is differentiable at $a \in U$ and g is differentiable at $b = f(a)$, then $h = g \circ f$ is differentiable at $x = a$ and

$$Dh(a) = Dg(F(a))Df(a).$$

(vi) **Corollary.** Let $f : U(\subset \mathbb{R}^n) \to V(\subset \mathbb{R}^m)$ and let $g : V \to \mathbb{R}^p$. If $f \in C^r(U)$ and $g \in C^r(V)$, then $g \circ f \in C^r(U)$.

(vii) Let $\mathscr{C} = \{x : (-\epsilon,\epsilon) \to \mathbb{R}^n : x \in C^1(-\epsilon,\epsilon), x(0) = a, \text{ and } \epsilon \in (0,\infty)\}$. Define an equivalence relation $\sim$ on $\mathscr{C}$ by $x(t) \sim y(t)$ is $x'(0) = y'(0)$, for $1 \le i \le n$. Then there exists a well-defined correspondence

$$\mathscr{C}/\sim \leftrightarrow V^n : [x(t)] \leftrightarrow (x_1'(0),\ldots,x_n'(0)), \qquad (*)$$

where V^n is vector space of dimension n over $\mathbb{R}$.

(viii) **Definition.** The correspondence in (*) above induces a vector space structure on $\mathscr{C}/\sim$ called the *tangent space of* $\mathbb{R}^n$ at a denoted by $T_a(\mathbb{R}^n)$.

(ix) **Definition.** A map $f : U(\subset \mathbb{R}^n) \to V(\subset \mathbb{R}^m)$is called a C^r*-diffeomorphism* if:

(a) f is a homeomorphism and

(b) both f and f^{-1} are of class C^r.

(x) Let $U, V, W \subset \mathbb{R}^n$ be open. Let $f : U \to V$ and $g : V \to W$ be onto mappings, and let $h = g \circ f$. If any two of these are diffeomorphisms, then so is the third.

(xi) **Theorem (Inverse Function Theorem).** Let $f : W(\subset \mathbb{R}^n) \to \mathbb{R}^n$ be a C^r mapping for some $r \geq 1$. If for $a \in W$, $Df(a)$ is non-singular, then there exists a neighborhood $U \ni a$ in W such that $V = f(U)$ is open and $f : U \to V$ is a C^r-diffeomorphism. In particular, if $y = f(x)$, then

$$Df^{-1}(y) = (Df(x))^{-1}.$$

(xii) **Corollary.** Let $f : W(\subset \mathbb{R}^n) \to \mathbb{R}^n$, where W is open. If $Df(a)$ is non-singular at each $a \in W$, then f is an open map.

(xiii) **Corollary.** A C^∞ map $f : W(\subset \mathbb{R}^n) \to \mathbb{R}^n$ is a diffeomorphism $W \to f(W)$ if and only if Df is non-singular at each $a \in W$.

(xiv) Let $f : U(\subset \mathbb{R}^n) \to \mathbb{R}^m$. Then the rank of $Df(x)$ is defined to be *rank of f at x*.

(xv) **Theorem (Rank Theorem).** Let $f : U_0(\subset \mathbb{R}^n) \to V_0(\subset \mathbb{R}^m)$ be a C^r-mapping and let rank of f be k at each $x \in U_0$. If $a \in U_0$ and $b = f(a)$, then there exists open sets $U \subset U_0$ and $V \subset V_0$ with $a \in U$ and $b \in V$, and there exists C^r-diffeomorphisms $g : U \to U'(\subset \mathbb{R}^m)$, $h : V \to V'(\subset \mathbb{R}^m)$ such that $h \circ f \circ g^{-1}(U') \subset V'$ and

$$h \circ f \circ g^{-1}(x_1, \ldots, x_n) = (x_1, \ldots, x_k, 0, \ldots, 0).$$

1.2 Smooth manifolds

1.2.1 Topological manifolds

(i) **Definition.** A topological space M is said to be *locally Euclidean of dimension n* if for each $p \in M$, there exists a neighborhood $U_p \ni p$ and a homeomorphism φ_p from U to an open set in $\mathbb{R}^n$, for some fixed n. Each pair (U_p, φ_p) is called a *coordinate neighborhood (or chart* of M.

(ii) **Definition.** An *topological n-manifold* (or a *topological manifold of dimension n*) is a topological space M with the following properties.

(a) M is Hausdorff.

(b) M is locally Euclidean of dimension n.

(c) M is second countable.

(iii) Examples of topological n-manifolds.

(a) An open subset of $\mathbb{R}^n$ is an n-manifold.

(b) The unit sphere S^2 is a 2-manifold.

(c) The torus $T^2 \approx S^1 \times S^1$ is a 2-manifold.

(d) The *real projective n-space* $\mathbb{R}P^n = \mathbb{R}^{n+1} - \{0\}/\sim$, where $x \sim y$, of $y = tx$, or equivalently, the space of all lines through the origin in $\mathbb{R}^{n+1}$ is an n-manifold.

(e) If M is a smoothly embedded 2-manifold in $\mathbb{R}^3$, then the *tangent bundle of M* defined by $T(M) := \bigcup_{p\in M} T_p(M)$ is a 4-manifold.

(iv) **Theorem.** A topological n-manifold M has the following properties.

(a) M is locally connected.

(b) M is locally compact.

(c) M is a countable union of compact sets (i.e. σ-compact).

(d) M is normal and metrizable.

(v) **Definition.** A *topological n-manifold with boundary* is a Hausdorff, second-countable space, where each $p \in M$ has a neighborhood $U \ni p$ such that U is homeomorphic via (a homeomorphism) φ to either:

(a) an open set of $\mathbb{H}^n - \partial\mathbb{H}^n$, where $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$, or

(b) an open set in $\mathbb{H}^n$ with $\varphi(p) \in \partial\mathbb{H}^n$.

(vi) Examples of manifolds with boundary.

(a) The annulus $S^1 \times I$ is a 2-manifold with two boundary components.

(b) The torus minus a disk is a 2-manifold with one boundary components.

(c) The sphere minus 3 (mutually disjoint) open disks (also known as a *pair of pants*) is a 2-manifold with three boundary components.

(vii) **Theorem (Classification of 2-manifolds or surfaces).**

(a) Every compact, connected, closed (without boundary), and orientable (resp. non-orientable) 2-manifold is homeomorphic to a sphere with $g \geq 0$ handles (resp. $g \geq 1$ crosscaps) attached.

(b) Every compact and connected 2-manifold with boundary is homeomorphic to a compact, connected, and closed 2-manifold with $b \geq 1$ mutually disjoint imbedded open disks removed.

1.2.2 Smooth manifolds

(i) **Definition.** Two coordinate neighborhoods (U_p, φ_p) and (U_q, φ_q) of a topological n-manifold M are said to be C^∞*-compatible* (or *smoothly compatible*) if $U_p \cap U_q \neq \emptyset$ implies that both $\varphi_p \circ \varphi_q^{-1}$ and $\varphi_q \circ \varphi_p^{-1}$ are diffeomorphisms.

(ii) **Definition.** A *differentible* (or C^∞ or *smooth*) structure on a topological manifold M is a family $\mathscr{U} = (U_\alpha, \varphi_\alpha)$ of coordinate neighborhoods of M that satisfies the following conditions.

(a) The U_α cover M.

(b) For any α, β, the coordinate neighborhoods $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are smoothly compatible.

(c) If (V, ψ) is a coordinate neighborhood that is smoothly compatible with every coordinate neighborhood in $\mathscr{U}$, then $(V, \psi) \in \mathscr{U}$.

If $\mathscr{U} = (U_\alpha, \varphi_\alpha)$ satisfies just (a) & (b), it is called an *atlas for M*, and if an atlas for M also satisfies (c) it is called a *maximal atlas for M*. Thus, a smooth structure on M is also known as a maximal atlas for M.

(iii) **Definition.** A *differentible* (or C^∞ or *smooth*) *n-manifold* is a topological n-manifold M together with a smooth structure on M.

(iv) **Theorem.** Let M be a Hausdorff and second-countable space. Let $\{U_\alpha, \varphi_\alpha\}$ be a covering of M by smoothly compatible coordinate neighborhoods. Then there exists a unique smooth structure on M containing these neighborhoods (called the *smooth structure determined by the* $\{U_\alpha, \varphi_\alpha\}$).

(v) Examples of differentiable manifolds.

(a) $\mathbb{R}^n$ with the standard topology is a differentiable manifold with a single coordinate neighborhood $(\mathbb{R}^n, id)$ determining a structure by Theorem 1.2.2 (iv).

(b) An n-dimensional vector space over $\mathbb{R}$ is a differentiable n-manifold. Consequently, the vector space $M_n(\mathbb{R})$ of $n \times n$ matrices over the reals is a differentiable n^2-manifold.

(c) An open subset of a differentiable n-manifold is also differentiable n-manifold.

(d) The general linear group $\mathrm{GL}(n, \mathbb{R})$ is a differentible n^2-manifold since $\mathrm{GL}(n, \mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$ under the determinant map $\det : M_n(\mathbb{R}) \to \mathbb{R}$.

(e) The unit sphere $S^2 \subset \mathbb{R}^3$ is a differentiable 2-manifold with the differentiable structure determined by $\{(U_i^\pm, \varphi_i^\pm) : 1 \le i \le 3\}$, where

$$U_i^\pm = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \pm x_i > 0\} \text{ and } \varphi_i^\pm(x_1, x_2, x_3) = \pi_i(x_1, x_2, x_3),$$

where π_i denotes the projection onto the coordinate plane with the unit vector e_i as the unit normal.

(f) The real projective n-space $\mathbb{R}P^n$ is a differentiable n-manifold with the structure determined by the coordinate neighborhoods $\{(U_i, \varphi_i) : 1 \le i \le n+1\}$, where

$$U_i = \{q(\bar{U}_i) : \bar{U}_i = \{x \in \mathbb{R}^{n+1} : x_i \neq 0\}\} \text{ and } q : \mathbb{R}^{n+1} \to \mathbb{R}P^n \text{ is the quotient map}\}$$

and $\varphi_i : U_i \to \mathbb{R}^n$ is defined by

$$\varphi_i(x_1, \dots x_{n+1}) = \left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{n+1}}{x_i}\right).$$

(g) The *Grassman manifold* $G(k,n)$ is defined to be the set of k-planes through the origin in $\mathbb{R}^n$. Let $F(k,n)$ denotes the set of k-frames (i.e. linearly independent sets of k elements) in $\mathbb{R}^n$. Define an equivalence relation $\sim$ on $F(k,n)$ by:

$$X \sim Y \iff \exists A \in \mathrm{GL}(n,\mathbb{R}) \text{ such that } Y = AX.$$

Then $G(k,n) \approx F(k,n)/\sim$. Hence, $G(k,n)$ is Hausdorff and the quotient map $\pi : F(k,n) \to G(k,n)$ is open. Given an ordered subset $J = (j_1,\dots,j_k)$ of $(1,2,\dots,n)$ and an $A \in M_{kn}(\mathbb{R})$, let $A_J = (a_{i\,j_\ell})_{1\le i,\ell\le k}$ be a $k \times k$ submatrix of A and A'_J be the complementary $k \times (n-k)$ matrix obtained by striking out the columns $j_1,\dots,j_k$ of A. Let $U_{\bar{J}}$ be the open set of $F(k,n)$ consisting of matrices for which A_J is nonsingular and let $U_J = \pi(U_{\bar{J}})$. Then $G(k,n)$ is a differentiable manifold with a differentiable structure determined by the coordinate neighborhoods $\{(U_J,\varphi_J)\}$, where $\varphi_J : U_J \to M_{k(n-k)}(\approx \mathbb{R}^{k(n-k)})$ defined by $\varphi(B) = B'_J$.

(vi) **Theorem.** If M is a differentiable m-manifold and N is a differentiable n-manifold, the $M \times N$ is a differentiable $(m+n)$-manifold.

1.2.3 Differentiable functions on smooth manifolds

(i) **Definition.** Let M be a smooth manifold. A map $f : W(\subset M) \to \mathbb{R}$, where W is open, is said to be C^∞ (or *smooth*) if each $p \in W$ lies in a coordinate neighborhood (U,φ) such that $f \circ \varphi^{-1}$ is C^∞ on $\varphi(W \cap V)$.

(ii) **Remark.** A C^∞ map as in the Definition above is continuous.

(iii) Examples of C^∞ maps.

(a) The coordinate projections of a coordinate neighborhood (U,φ) defined by $x_i(q) = \pi_i(\varphi(q))$, for each $q \in U$ are C^∞.

(b) If $F \in C^\infty(W)$ and $V \subset W$ is open, then $F|_W \in C^\infty(W)$.

(c) If $W = \cup_\alpha V_\alpha$, where V_α is open and $F \in C^\infty(V_\alpha)$ for each α, then $f \in C^\infty(W)$.

(d) If $f \in C^\infty(W)$ and (V,ψ) is a coordinate neighborhood such that $V \cap W \neq \emptyset$, then $f \circ \psi^{-1} \in C^\infty(\psi(V \cap W))$.

(iv) **Definition.** Let M and N be smooth manifold, and let $F : W(\subset M) \to N$, where W is open. Then f is said to be a C^∞ (or *smooth*) *mapping* if for each $p \in W$, there exists coordinate neighborhoods (U, φ) of p and (V, ψ) of $f(p)$ with $f(U) \subset V$ such that $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \varphi(V)$ is C^∞.

(v) **Remark.** C^∞ mappings satisfy the following properties.

(a) They are continuous.

(b) The constructions in Examples (b)-(d) also hold true in the setting of C^∞ mappings.

(vi) **Definition.** Let M and N be smooth manifolds. A C^∞ mapping $f : M \to N$ is said to be a *diffeomorphism* if f is a homeomorphism and f^{-1} is C^∞.

(vii) **Remark.**

(a) The relation of diffeomorphism between smooth manifolds is an equivalence relation.

(b) Smooth manifolds with the same underlying topological manifolds but incompatible C^∞ structures can be diffeomorphic. For example, consider the smooth structure $(\mathbb{R}, f)$ on $\mathbb{R}$, where $f(t) = t^3$. Note that $f \in C^\infty(\mathbb{R})$ and is a homeomorphism, but not a diffeomorphism since $f^{-1}(t) = \sqrt[3]{t} \notin C^1(\mathbb{R})$. Furthermore, the smooth structures $(\mathbb{R}, id)$ and $(\mathbb{R}, f)$ on $\mathbb{R}$ are not C^∞ compatible. However, $\mathbb{R}$ with these two structures are diffeomorphic.

(c) It is a non-trivial fact that a topological manifold M can have non-diffeomorphic C^∞ structures. Milnor gave examples of non-diffeomorphic C^∞ structures on S^7.

(viii) **Definition.** Let $F : N \to M$ be a differentiable mapping of smooth manifolds and let $p \in N$. Let (U, φ) and (V, ψ) be coordinate neighborhoods of p and $f(p)$ such that $f(U) \subset V$. Then the *rank of f at p* is defined as the rank of $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$.

(ix) **Remark.** The rank of f at p is the rank of the Jacobian matrix of $\psi \circ f \circ \varphi^{-1}$ at $\varphi(p)$.

(x) **Theorem (Rank Theorem).** Let $F : N \to M$ be a differentiable mapping of smooth manifolds and let $p \in N$. Let $\dim(M) = m$, $\dim(N) = n$, and rank$(f) = k$ at each point of N. Then there exists coordinate neighborhoods (U, φ) and (V, ψ) of p and $f(p)$ with $f(U) \subset V$ such that:

(a) $\varphi(p) = 0 \in \mathbb{R}^n$, $\varphi(U) = C^n_\epsilon(0)$,

(b) $\psi(f(p)) = 0 \in \mathbb{R}^m$, $\varphi(V) = C^m_\epsilon(0)$, and

(c) $(\psi \circ f \circ \varphi^{-1})(x_1, \dots, x_n) = (x_1, \dots, x_k, 0, \dots, 0)$.

(xi) **Corollary.** If $f : N \to M$ is a diffeomorphism, then $\dim(M) = \dim(N) = \text{rank}(f)$.

(xii) **Definition.** A C^∞ mapping $f : N \to M$ between smooth manifolds is said to be an *immersion* (resp. *submersion*) if $\text{rank}(f) = \dim(N)$ (resp. $\text{rank}(f) = \dim(M)$).

(xiii) **Remark.**

(a) Since $\text{rank}(f) \le \max(\dim(M), \dim(N))$ at every point, it follows that if f is an immersion (resp. submersion), then $\dim(M) \le \dim(N)$ (resp. $\dim(M) \ge \dim(N)$).

(b) If $f : N \to M$ is an injective immersion, then using the correspondence $N \leftrightarrow f(N)$, $f(N)$ can be endowed with a topology and a C^∞ structure from N under which $f : N \to f(N)$ is a diffeomorphism.

(xiv) **Definition.** Let $f : N \to M$ is an injective immersion. Then $f(N)$ is called an *immersed submanifold* of M.

(xv) **Remark.**

(a) Immersions need not be injective.

(b) Even when injective, an immersion need not define a homeomorphism onto its image.

(xvi) **Definition.** An injective immersion $f : N \to M$ that defines a homeomorphism $\tilde{f} : N \to f(N)$ onto its image is called an *imbedding.*

(xvii) Example of immersions.

(a) The map $f : \mathbb{R} \to \mathbb{R}^3$ be defined by $f(t) = (\cos(2\pi t), \sin(2\pi t), t)$ is an imbedding whose image is an infinite helix on the unit infinite cylinder with the z-axis as axis.

(b) The map $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(t) = (\cos(2\pi t), \sin(2\pi t))$ a (non-injective) immersion whose image is the unit circle centered at the origin.

(c) The map $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(t) = (2\cos(t - \pi/2), \sin(2t - \pi))$ a (non-injective) immersion whose image is a figure-eight curve (also known as a *lemniscate*).

(d) The map $f \circ g$, where $g(t) = \pi + 2\tan^{-1}(t)$ and f as in example (c) above, is an injective immersion that is not an imbedding.

(xviii) **Theorem.** Let $f : N \to M$ be an immersion. Then at each $p \in N$ there exists a neighborhhod $U \ni p$ such that $f|_U$ is an imbedding of U in M.

1.2.4 Submanifolds

(i) **Definition.** A subset N of a smooth m-manifold M is said to have the *n-submanifold property* if each $p \in N$ has a coordinate neighborhood (U, φ) on M such that: .

(a) $\varphi(p) = 0 \in \mathbb{R}^n$,

(b) $\varphi(U) = C^m_\epsilon(0)$, and

(c) $\varphi(U \cap N) = \{x \in C^m_\epsilon(0) : x_{n+1} = \ldots = x_m = 0\}$.

If an $N \subset M$ satisfies this property, then any coordinate neighborhood satisfying (a) - (c) above is called a *preferred coordinate neighborhood.*

(ii) **Lemma.** Let M be a smooth m-manifold, and let $N \subset M$ have the smooth n-submanifold property. Then:

(a) N with the subspace topology is a topological n-manifold.

(b) Each coordinate neighborhood (U, φ) on M defines a coordinate neighborhood $(U \cap N, \pi \circ \varphi|_V)$ on M and these coordinate neighborhoods define an induced C^∞ structure on N.

(c) Relative to the induced structure above, the inclusion $N \hookrightarrow M$ is an imbedding.

(iii) **Definition.**A *regular submanifold* N of a smooth m-manifold M is a subspace of M with the n-submanifold property and with C^∞ structure that the corresponding preferred coordinate neighborhoods determine on it.

(iv) **Theorem.** Let N' and M be smooth smooth manifolds of dimensions n and n respectively, and let $f : N' \to M$ be an imbedding. Then:

   (a) $N = f(N')$ has the n-submanifold property and is hence a regular submanifold of M, and

   (b) f defines a diffeomorphism $\tilde{f} : N' \to N$ onto its image.

(v) **Theorem.** Let N' and M be smooth manifolds of dimensions n and m respectively, and $f : N' \to M$ is an injective immersion. If N is compact, then $N = f(N')$ is a regular n-submanifold. Consequently, a compact submanifold of M is regular.

(vi) **Theorem (Regular Value Theorem).** Let N and M be smooth manifolds of dimensions n and m respectively, and $f : N \to M$ be a C^∞ mapping. If f has constant rank k on N, then for any $q \in f(N)$, $f^{-1}(q)$ is a closed regular submanifold of N of dimension $n-k$.

(vii) **Corollary.** Let N and M be smooth manifolds of dimensions n and m respectively, and $f : N \to M$ be a C^∞ mapping. If $m \leq n$ and $\operatorname{rank}(f) = m$ at each point of $A = f^{-1}(a)$, then A is a closed regular submanifold of M of dimension $n-m$.

(viii) Example of regular submanifolds.

   (a) The smooth map $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i^2$. has constant rank 1 on $\mathbb{R}^n \setminus \{0\}$. Thus, by the Regular Value Theorem, the unit sphere $S^{n-1} = f^{-1}(0)$ is a submanifold of $\mathbb{R}^n \setminus \{0\}$, and hence $\mathbb{R}^n$ of dimension $n-1$.

   (b) The smooth map $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x_1, x_2, x_3) = \left(a - \sqrt{x_1^2 + x_2^2}\right)^2 + x_3^2,$$

   where $a > b > 0$, has constant rank 1 at each point of the torus $f^{-1}(b^2)$. Thus, by the Corollary to the Regular Value Theorem, it follows that the torus is a submanifold of $\mathbb{R}^3$ of dimension 2.

1.3 Lie groups and their actions on manifolds

1.3.1 Lie groups

(i) **Definition.** Let G be a group and a smooth manifold. Then G is *Lie group* if the group operation $G \times G \to G : (g,h) \mapsto gh$ and the inverse mapping $G \to G : g \mapsto g^{-1}$ are C^∞ mappings.

(ii) Examples of Lie groups.

(a) The general linear group $\mathrm{GL}(n,\mathbb{R})$ is a Lie groups with respect to matrix multiplication.

(b) $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is a Lie group with respect to complex multiplication. Note that C^* is a smooth manifold with a differentiable structure comprising single coordinate neighborhood (U,φ), where $U = C^*$ and $\varphi : C^* \to \mathbb{R}^2$ defined by $\varphi(x+iy) = (x,y)$.

(iii) **Lemma.** Let $f : A \to M$ be a C^∞ mapping of C^∞ manifolds. If $f(A) \subset N$, where N is a regular submanifold, then f is a C^∞ mapping onto N.

(iv) **Theorem.** Let G be a Lie group and $H < G$ be a regular submanifold. Then with its differentiable structure as a submanifold, H is a Lie group.

(v) **Theorem.** If G_1 and G_2 are Lie groups, then $G_1 \times G_2$ is a Lie group with the C^∞ structure coming from the Cartesian product of the manifolds.

(vi) More examples of Lie groups

(a) By Theorem 2.1(iv) above, $S^1 \subset C^*$ is a Lie group. Consequently, by Theorem 2.1 (v), the n torus $T^n = \prod_{i=1}^n S^1$ is a Lie group.

(b) The *special linear group* $\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) : \det(A) = 1\}$ is a Lie group of dimension $n^2 - 1$. This follows from the Regular Value Theorem and Theorem 2.1(iv) since the C^∞ mapping $\det : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}^*$ has constant rank 1 and $\mathrm{SL}(n,\mathbb{R}) = \det^{-1}(1)$.

(c) The *orthogonal group* $\mathrm{O}(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) : AA^T = I_n\}$ is a Lie group of dimension $n(n-1)/2$. This follows from the Regular Value Theorem and Theorem 2.1(iv) since the C^∞ mapping $f : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ defined by $f(A) = AA^T$ has constant rank $n(n+1)/2$ and $\mathrm{O}(n,\mathbb{R}) = f^{-1}(I_n)$.

(vii) **Definition.** Let G_1 and G_2 be Lie groups. We call an $f: G_1 \to G_2$ a *Lie group homomorphism* if:

(a) f is a homomorphism and

(b) f is a C^∞ mapping.

(viii) Example of Lie group homomorphisms.

(a) The map $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^*$ is a Lie group homomorphism.

(b) The (covering) map $p : \mathbb{R} \to S^1$ defined by $p(x) = e^{2\pi i x}$ is a Lie group homomorphism. By extension, $p^n : \mathbb{R}^n \to T^n$ is a Lie group homomorphism.

(c) Consider the covering map $p^2 : \mathbb{R}^2 \to T^2$ from the preceding example and a line $L_\alpha \subset \mathbb{R}^2$ through the origin of irrational slope α given by $L_\alpha = \{(x, \alpha x) : x \in \mathbb{R}\}$. Then $p^2(L_\alpha)$ is a dense subset of T^2. Moreover, $p^2|_{L_\alpha} : L_\alpha \to T^2$ is an injective immersion. Thus, $f(L_\alpha)$ is an immersed submanifold of T^2.

Moreover, if $g : \mathbb{R} \to \mathbb{R}^2$ is defined by $g(t) = (t, \alpha t)$, then $p^2 \circ g : \mathbb{R} \to T^2$ is a Lie group homomorphism and $(p^2 \circ g)(\mathbb{R}) = p^2(L_\alpha)$ is Lie group. However, $p^2(L_\alpha)$ is neither closed or a regular submanifold of T^2.

(ix) **Theorem.** If $f : G_1 \to G_2$ is a Lie group homomorphism, then:

(a) $\mathrm{rank}(f)$ is constant and

(b) $\ker f$ is a closed regular submanifold of G_1.

(x) **Theorem.** If H is a regular submanifold and a subgroup of a Lie group G, then H is a closed subset of G.

1.3.2 Lie group actions

(i) **Definition.** Let G be a Lie group, and X be a smooth manifold. Then G *acts smoothly on* X (in symbols $G \curvearrowright X$) is there exists a C^∞ mapping $\theta : G \times X \to X$ satisfying the following conditions.

(a) If $e \in G$ is the identity, then $\theta(e, g) = g$, for all $g \in G$.

(b) If $g_1, g_2 \in G$, then $\theta(g_1, \theta(g_2, x)) = \theta(g_1 g_2, x)$, for all $x \in X$.

(ii) **Notation**

(a) We often write $\theta(g,x)$ in the definition above simply as $g \cdot x$ or gx.

(b) For a fixed $g \in G$, we denote by θ_g, the mapping $x \mapsto gx$, for all $x \in G$.

(iii) **Remark.** $G \curvearrowright X$ if and only if the map $G \to \text{Diffeo}(X)$ defined by $g \mapsto \theta_g$ is a homomorphism.

(iv) **Definition.** Let G be a Lie group, and X be a smooth manifold. Then a smooth action of G on X is *effective (or faithful)* if the homomorphism $g \to \theta_g$ is injective.

(v) Example of Lie group actions.

(a) Let H and G be Lie groups, and $\psi : H \to G$ a Lie group homomorphism. Then $\theta : H \times G \to G$ defined by $\theta(h,x) = \psi(h)(x)$ is a smooth action.

(b) The natural action of $\text{GL}(n,\mathbb{R})$ on $\mathbb{R}^n$ is a smooth action which has a unique fixed point $\{0\}$. Note that this is a transitive action on $\mathbb{R}^n \setminus \{0\}$.

(c) If $H < \text{GL}(n,\mathbb{R})$ and the inclusion $H \hookrightarrow \text{GL}(n,\mathbb{R})$ is an immersion or an imbedding, then restricted action of H on $\mathbb{R}^n$ is smooth. For example, the restricted action of the subgroup

$$H = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \in \text{GL}(2,\mathbb{R}) : a > 0 \right\},$$

which is a two-dimensional regular submanifold of $\text{GL}(2,\mathbb{R})$, on $\mathbb{R}^2$, is smooth.

(d) The Lie group $G = \text{Isom}(\mathbb{R}^n) \cong \text{O}(n,\mathbb{R}) \times \mathbb{R}^n$ of rigid motions in $\mathbb{R}^n$ acts smoothly on $\mathbb{R}^n$ and this action is given by $\theta : G \times \mathbb{R}^n \to \mathbb{R}^n$, where $\theta((A,b),x) = Ax + b$.

(e) The group $\text{GL}(n,\mathbb{R})$ acts transitively on the set $\mathscr{B}$ of bases of $\mathbb{R}^n$ (also known as the space of *n-frames* of $\mathbb{R}^n$). Given a basis $f = \{f_1, \dots, f_n\}$ of $\mathbb{R}^n$, there exists a unique matrix in $\text{GL}(n,\mathbb{R})$ that maps the standard basis $e = \{e_1, \dots, e_n\}$ to f. Thus, there is a correspondence $\mathscr{B} \leftrightarrow \text{GL}(n,\mathbb{R})$, which is a diffeomorphism. Hence, $\mathscr{B}$ is a smooth manifold and the action of $\text{GL}(n,\mathbb{R})$ on $\mathscr{B}$ is smooth.

(f) The Lie group $\text{O}(n,\mathbb{R})$ acts on $\mathbb{R}^n$ smoothly and the orbits of this action are concentric spheres centered at the origin. Thus, $\mathbb{R}^n/\text{O}(n) \approx [0,\infty)$ which is not a smooth manifold.

(vi) **Theorem.** Let G be a Lie group and $H < G$ in an algebraic subgroup. Then the map $G \to G/H$ is continuous and open. Furthermore, G/H is Hausdorff if and only if H is closed.

(vii) **Definition.** The action of a Lie group G on a manifold X is said to be free if $g \cdot x = x$ for any $g \in G$ and $x \in X$, then it would imply that $g = e$.

1.3.3 Discrete groups and properly discontinuous actions

(i) **Definition.** A *discrete group* Γ is a countable group with the discrete topology.

(ii) **Remark.** A discrete group is a zero-dimensional Lie group.

(iii) **Definition.** A discrete group Γ is said to act *properly discontinuously* on a manifold $\tilde{M}$ if the action is C^∞ satisfying the following conditions.

 (a) Each $x \in \tilde{M}$ has a neighborhood $U \ni x$ such that $\{h \in \Gamma : h(U) \cap U \neq \emptyset\}$ is finite.

 (b) If $x, y \in \tilde{M}$ are not in the same orbit, then there exists neighborhoods $U \ni x$ and $V \ni y$ such that $U \cap \Gamma(V) = \emptyset$.

(iv) **Remark.** If a discrete group Γ acts properly discontinuously on a manifold $\tilde{M}$, then $\tilde{M}/\Gamma$ is Hausdorff.

(v) **Definition.** Let $\tilde{M}$ and M be smooth manifolds, and let $\pi : \tilde{M} \to M$ be a smooth and surjective map. The π is said to be a *covering map* if at each $p \in M$ there exists a connected neighborhood $U \ni p$ such that:

 (a) $\pi^{-1}(U) = \sqcup_\alpha V_\alpha$, where V_α is open, and

 (b) for each α, $\pi|_{V_\alpha} : V_\alpha \to U$ is a diffeomorphism.

 A neighborhood U satisfying properties (a) and (b) is a called an *evenly covered neighborhood.* If there exists a covering map $\pi : \tilde{M} \to M$, then the manifold $\tilde{M}$ is said to be a *covering manifold* of M.

(vi) **Theorem.** Let Γ be discrete group that acts freely and properly discontinuously on a manifold $\tilde{M}$, there exists a unique C^∞ structure on $M = \tilde{M}/\Gamma$ such that $\tilde{M}$ is a covering manifold of M.

(vii) **Remark.** The rank of a covering map $\pi : \tilde{M} \to M$ equals $\dim(M) = \dim(\tilde{M})$ since it is a local diffeomorphism.

(viii) **Lemma** Let G be a Lie group and Γ an algebraic subgroup of G. Then there exists a neighborhood $U \ni e$ such that $\Gamma \cap U = \{e\}$ if and only if Γ is a discrete subspace, in which case $\overline{\Gamma} = \Gamma$.

(ix) **Theorem**. Any discrete subgroup Γ of a Lie group G acts freely and properly discontinuously on G by left multiplication.

(x) **Corollary.** If Γ is a discrete subgroup of a Lie group G, then G/Γ is a C^∞ manifold and $\pi : G \to G/\Gamma$ is smooth.

(xi) **Theorem.** Let $\pi : \tilde{M} \to M$ be the covering of a smooth manifold M by a connected smooth manifold $\tilde{M}$. Then the *group of deck transformations*

$$\mathrm{Deck}(\pi) := \{f \in \mathrm{Diffeo}(\tilde{M}) : f \circ \pi = \pi\}$$

acts freely and properly discontinuously on $\tilde{M}$ and the quotient map $\pi_1 : \tilde{M} \to \tilde{M}/\mathrm{Deck}(p)$ is a covering map. If $\mathrm{Deck}(\pi)$ acts transitively on the fibers of π, then π_1 and $\tilde{M}/\mathrm{Deck}(\pi)$ can be naturally identified with π and M, respectively.

(xii) Examples of discrete group actions.

(a) The action $\mathbb{Z}_2 \times S^n \to S^n$ defined by $([1], x) \mapsto -x$ is a free and properly discontinuous action and under this action, $S^n/\mathbb{Z}_2 \approx \mathbb{R}P^n$. Thus, by Theorem 1.3.3 (xi), it follows that the quotient map $S^n \to \mathbb{R}P^n$ is a covering map and S^n is a covering manifold of $\mathbb{R}P^n$.

(b) The action $\mathbb{Z}^n \times \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$((k_1, \dots, k_n), (x_1, \dots, x_n)) \mapsto (x_1 + k_1, \dots, x_n + k_n)$$

is a free and properly discontinuous action and under this action, $\mathbb{R}^n/\mathbb{Z}^n \approx T^n = \prod_{i=1}^n S^1$. Thus, by Theorem 1.3.3 (xi), it follows that the quotient map $\mathbb{R}^n \to T^n$ is a covering map and $\mathbb{R}^n$ is a covering manifold of T^n.

2 Vector fields on manifolds

2.1 Tangent space at a point on a manifold

(i) **Definition.** Let M be a smooth manifold. Given any $p \in M$ consider the collection

$$C_p = \{f : U(\subset M) \to \mathbb{R} : f \in C^\infty(U), U \text{ is open, and}$$
$$U \text{ contains a neighborhood of p}\}.$$

Define an equivalence relation on C_p given by $f \sim g$ if f and g agree on some neighborhood of p. Then $\mathscr{C}^\infty(p) := C_p / \sim$ is called *algebra of germs of C^∞ functions at p.*

(ii) **Remark.** Given a coordinate neighborhood (U, φ) of $p \in M$, the induced algebra homomorphism $\varphi^* : \mathscr{C}^\infty(\varphi(p)) \to \mathscr{C}^\infty(p)$ defined by $\varphi^*(f) = f \circ \varphi$ is an isomorphism of algebras of germs of C^∞ functions.

(iii) **Definition.** The *tangent space $T_p(M)$ to M at p* is the set of all mappings $\{\mathscr{C}^\infty(p) \to \mathbb{R}\}$ satisfying the following conditions for all $\alpha, \beta \in R$, $f, g \in \mathscr{C}^\infty(p)$, and $X_p, Y_p \in T_p(M)$.

(a) $X_p(\alpha f + \beta g) = \alpha X_p(f) + \beta X_p(g)$. (Linearity)

(b) $X_p(fg) = X_p(f)g(p) + f(p)X_p(g)$. (Leibnitz rule)

(c) The vector space operations:

(1) $(X_p + Y_p)(f) = X_p(f) + Y_p(f)$.

(2) $(\alpha X_p)(f) = \alpha X_p(f)$.

A *tangent vector to M at $p \in M$* is any $X_p \in T_p(M)$.

(iv) **Theorem.** Let $F : M \to N$ be a C^∞ map of smooth manifolds, and let $p \in M$. Then:

(a) The map $F^* : \mathscr{C}^\infty(f(p)) \to \mathscr{C}^\infty(p)$ defined by $F^*(f) = f \circ F$ is an algebra homomorphism.

(b) The homomorphism F^* the induces a dual homomorphism $F_* : T_p(M) \to T_{F(p)}(N)$ defined by $F_*(X_p)(f) = X_p(F^*(f))$.

(v) **Corollary.**

(a) If $F : M \to M$ is the identity map on a smooth manifold M, then both F^* and F_* are identity isomorphisms.

(b) If $H = G \circ F$ is a composition of C^∞ maps on smooth manifolds, then $H^* = F^* \circ G^*$ and $H_* = G_* \circ F_*$.

(vi) **Corollary.** If $F : M \to N$ is a diffeomorphism of smooth manifold M onto an open set of a smooth manifold N, then each $p \in M$, the homomorphism $F_* : T_p(M) \to T_{F(p)}(N)$ is an isomorphism.

(vii) **Remark.** Let M be a smooth n-manifold and let (U, ϕ) be a coordinate neighborhood of $p \in M$. Then by Corollary 2.1 (vi), ϕ induces an isomorphism $\varphi_* : T_p(M) \to T_{\varphi(p)}(\mathbb{R}^n)$ at each $p \in U$. Consequently, $\varphi_* : T_{\varphi(p)}(\mathbb{R}^n) \to T_p(M)$ is an isomorphism and for $1 \le i \le n$, the images $E_{ip} = \varphi_*^{-1}\left(\frac{\partial}{\partial x_i}\right)$ of the natural basis at $\varphi(p) \in \varphi(U)$ determines a basis of $T_p(M)$.

(viii) **Corollary.** Let M be a smooth n-manifold.

(a) To each coordinate neighborhood $(U\varphi)$ of a smooth n-manifold M, there corresponds a natural basis $E_{1p}, \ldots, E_{np}$ of $T_p(M)$, for all $p \in U$. Consequently,

$$\dim(T_p(M)) = n = \dim(M).$$

(b) Let f be a C^∞ function defined on a neighborhood of p and let $\hat{f} = f \circ \varphi^{-1}$ be its expression in local coordinates relative to (U, φ). Then:

$$E_{ip}(f) = \left(\frac{\partial \hat{f}}{\partial x_i}\right)_{\varphi(p)}.$$

(c) In particular, if $x_i(q)$ is the i^{th} coordinate function, then:

$$X_p = \sum_{i=1}^{n} (X_p(x_i)) E_{ip}.$$

(ix) **Remark.** Let M be a smooth n-manifold and let (U, ϕ) be a coordinate neighborhood of $p \in M$. Since $E_{ip} = \varphi_*^{-1}\left(\frac{\partial}{\partial x_i}\right)$, we have:

$$E_{ip}(f) = \varphi_*^{-1}\left(\frac{\partial}{\partial x_i}\right)(f) = \frac{\partial}{\partial x_i}(f \circ \varphi^{-1})\bigg|_{x=\varphi(p)}.$$

In particular , if $f(q) = x_i(q)$ and $X_p = \sum_{j=1}^{n} \alpha_j E_{jp}$, then we have

$$X_p(x_i) = \sum_{j=1}^{n} \alpha_j (E_{jp}(x_i)) = \sum_{j=1}^{n} \alpha_j \left(\frac{\partial x_i}{\partial x_j}\right)_{\varphi(p)} = \alpha_i.$$

(x) **Theorem.** Let M and N be smooth manifolds of dimensions m and n, respectively, and let $F : M \to N$ be a smooth map. Let (U,φ) and (V,ψ) be coordinate neighborhoods such that $F(U) \subset V$, and in these coordinates let F be given by $y_i = f(x_, \dots, x_n)$, $1 \le i \le m$. Let p be a point with coordinates $a = (a_1, \dots, a_n)$, $E_{ip} = \varphi_*^{-1}\left(\frac{\partial}{\partial x_i}\right)$, $1 \le i \le m$ be a basis of $T_p(M)$, and $\tilde{E}_{jF(p)} = \varphi_*^{-1}\left(\frac{\partial}{\partial y_j}\right)$, $1 \le j \le n$, be a basis of $T_{F(p)}(N)$. Then:

(a) For $1 \le i \le n$, we have:

$$F_*(E_{ip}) = \sum_{j=1}^{m} \left(\frac{\partial y_j}{\partial x_i}\right)_a \tilde{E}_{jF(p)}.$$

(b) In terms of components, if $X_p = \sum_{i=1}^{n} \alpha_i E_{ip}$ and $F_*(X_p) = \sum_{j=1}^{m} \tilde{E}_{jF(p)}$, then for $1 \le j \le m$, we have:

$$\beta_j = \sum_{i=1}^{n} \alpha_i \left(\frac{\partial y_j}{\partial x_i}\right)_a.$$

(xi) **Remark.** Let M be a smooth submanifold of N, and let $F : M \to N$ be an immersion or an inclusion of M into N. Then we have rank$(F) = \dim(M)$, and hence, $F_* : T_p(M) \to T_p(N)$ is injective (i.e, an isomorphism onto its image). Consequently, $T_p(M)$ can be identified with a subspace of $T_p(N)$.

(xii) Applications of Theorem 2.1 (x).

(a) *Change of basis formula for $T_p(M)$*. We apply Theorem 2.1 (x) to the maps $F = \tilde{\varphi} \circ \varphi^{-1}$ and F^{-1}, which give the change of coordinates between the coordinate neighborhoods (U,φ) and $(\tilde{U},\tilde{\varphi})$ in $U \cap \tilde{U}$ on M. For $p \in U \cap \tilde{U}$, let $E_{ip} = \varphi_*^{-1}\left(\frac{\partial}{\partial x_i}\right)$ and $\tilde{E}_{ip} = \tilde{\varphi}_*^{-1}\left(\frac{\partial}{\partial \tilde{x}_i}\right)$ be the bases of $T_p(M)$ corresponding to (U,φ) and $(\tilde{U},\tilde{\varphi})$, respectively. Then we have:

$$\begin{aligned} E_{ip} &= \sum_k \left(\frac{\partial \tilde{x}_k}{\partial x_i}\right)_{\varphi(p)} \tilde{E}_{kp}, 1 \le i \le n, \text{ and} \\ \tilde{E}_{jp} &= \sum_\ell \left(\frac{\partial x_\ell}{\partial \tilde{x}_j}\right)_{\tilde{\varphi}(p)} \tilde{E}_{\ell p}, 1 \le \ell \le n. \end{aligned}$$

In particular, if:

$$X_p = \sum_{i=1}^n \alpha_i E_{ip} = \sum_{j=1}^n \beta_j \tilde{E}_{jp},$$

then:

$$\alpha_i = \sum_{j=1}^n \beta_j \frac{\partial x_i}{\partial \tilde{x}_j} \text{ and } \beta_j = \sum_{i=1}^n \alpha_i \frac{\partial \tilde{x}_j}{\partial x_i}.$$

(b) *Tangent to a space curve.* Let $F:(a,b)\to N$ be a C^∞ curve. Then for $t_0 \in (a,b)$, we have $\left(\frac{d}{dt}\right)_{t_0}$ is a basis for $T_{t_0}(M)$. If $p = F(t_0)$ and $f \in \mathscr{C}^\infty(p)$, then

$$F_*\left(\frac{d}{dt}\right)(f) = \left(\frac{d}{dt}(f\circ F)\right)_{t_0},$$

which is called the to the curve $F(t)$ at p. In particular, if (U,φ) are the coordinates around p, then in local coordinates F is given by:

$$\hat{F}(t) = (\varphi\circ F)(t) = (x_1(t),\ldots,x_n(t),$$

where each x_i is a function on U. To simplify notation, we write $x_i(t) = (x_i \circ F)(f)$, and we have:

$$F_*\left(\frac{d}{dt}\right)(x_i) = \left(\frac{dx_i}{dt}\right)_{t_0} := \dot{x}_i(t_0).$$

Applying Theorem 2.1 (x) (with $E_{1p} = \frac{d}{dt}$ and the Es replaces with $\tilde{E}$s), we have:

$$F_*\left(\frac{d}{dt}\right) = \sum_{i=1}^n \dot{x}_i(t) E_{ip}.$$

When $N = \mathbb{R}^n$, $\frac{d}{dt}$ is the velocity vector at the point $p = (x_1(t_0),\ldots,x_n(t_0))$ whose components (at p) are $(\dot{x}_1(t_0),\ldots,\dot{x}_n(t_0))$. This is the vector $v_p \in T_p(\mathbb{R}^n)$ with initial point $p = x(t_0)$ and terminal point

$$(x_1 + \dot{x}_1(t_0),\ldots,x_n + \dot{x}_n(t_0)).$$

If $\text{rank}_{t_0}(F) = 1$, then F_* is an isomorphism onto its image, and we identify the tangent space to the image curve at p with the subspace of $T_p(\mathbb{R}^n)$ spanned by v_p. On the other hand, if $\text{rank}_{t_0}(F) = 0$, then $F_*\left(\frac{d}{dt}\right) = 0$.

2.2 Vector fields

(i) **Definition.** A *vector field X of class C^r* on a smooth manifold M is a mapping $X : M \to T(M) = \cup_{p\in M} T_p(M)$ that assigns to each $p \in M$ a vector $X_p \in T_p(M)$ whose components in the local frames $\{E_{1p}, \dots, E_{np}\}$ of any coordinate neighborhood (U, φ) of p are of class C^r on U.

(ii) Examples of vector fields.

(a) The unit gravitational vector field G on $M = \mathbb{R}^3 - \{0\}$ of an object of unit mass at 0 is a smooth mapping $G : M \to T(M)$ defined by

$$G(p) = \sum_{i=1}^{3} -\frac{x_i}{r^3} \frac{\partial}{\partial x_i}\bigg|_p ,$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

(b) Given any coordinate neighborhood (U, φ) on a smooth manifold M, for each $1 \le i \le n$, $E_i = \varphi_*^{-1}\left(\frac{\partial}{\partial x_i}\right)$ having component δ_{ij} is a C^∞ vector field on U. The set $\{E_1, \dots, E_n\}$ form a basis for $T_p(M)$ at each $p \in U$ called the coordinate frame associated to (U, φ).

(c) It is known there non-vanishing C^∞ vector fields on even-dimensional spheres, while odd-dimensional spheres have at least one non-vanishing vector field. For example on

$$S^3 = \{(x_1, x_2, x_3, x_4) : \sum_{i=1}^{4} x_i^2 = 1\},$$

there are three mutually perpendicular unit vector fields given by:

$$\begin{aligned} X &= -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_4} \\ Y &= -x_3 \frac{\partial}{\partial x_1} - x_4 \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_4} \\ Z &= -x_4 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_4} \end{aligned}$$

(iii) **Definition.** A smooth manifold M with a C^∞-vector field of bases is said to be *parallelizable*.

(iv) **Lemma**. Let N be a submanifold of M, and let X be a C^∞-vector field on M such that for each $p \in N$, $X_p \in T_p(N)$. Then $X|_N$ is a C^∞-vector field.

(v) **Remark.** Let N, M be smooth manifolds, and let $F : N \to M$ be a smooth map. Then given a vector field X on N, $F_*(X_p)$ is a vector at $T_{F(p)}(M)$. However, this process does not in general induce a vector field on M. This is because:

   (a) F need not be surjective and

   (b) even when F is surjective, there might exist $p_1, p_2 \in N$ with $F(p_i) = q$ such that $F_*(X_{p_1}) \neq F_*(X_{p_2})$.

(vi) **Definition.** Let N, M be smooth manifolds, and let $F : N \to M$ be a smooth map. Suppose there exists a vector field Y on M such that for each $q \in M$ and $p \in F^{-1}(q) \in N$, we have $F_*(X_p) = X_q$. Then we say that the vector fields X and Y are *F-related* and we write $Y = F_*(X)$

(vii) **Theorem.** If $F : N \to M$ is a diffeomorphism, then each vector field X on N is F-related to a uniquely determined vector field Y on M.

(viii) **Definition.** Let M be a smooth manifold and $F : M \to M$ be a diffeomorphism. Then X is said to be *F-invariant* if $F_*(X) = X$.

(ix) **Definition.** Let G be a Lie group and for a fixed $g \in G$, let $L_g : G \to G$ be left multiplication by g, that is, $L_g(h) = gh$, for all $h \in G$. Then a vector field X on G is a said to *left-invariant (or invariant under left translations)* if $(L_g)_*(X) = X$ for all $g \in G$.

(x) **Theorem.** Let G be a Lie group and $e \in G$ be the identity element. Then each $X_e \in T_e(G)$ determines a unique C^∞-vector field X on G that is left-invariant. In particular, G is parallelizable.

(xi) **Corollary.** Let G_1 and G_2 be Lie groups and $F : G_1 \to G_2$ be a homomorphism. Then to each left-invariant vector field X on G_1, there exists a uniquely determined left-invariant vector field Y on G_2 such that $F_*(X) = Y$.

2.3 Flows on manifolds

(i) **Definition.** Let $\theta : \mathbb{R} \to \text{Diffeo}(M)$ defined by $\theta(t) = \theta_t$ be a C^∞-action on a smooth manifold M. Then θ defines a C^∞-vector field X^θ on M given by $X^\theta(p) = X^\theta_p$, where $X_p\theta : \mathscr{C}^\infty(p) \to \mathbb{R}$ is defined by

$$X^\theta_p(f) = \lim_{\Delta t \to 0}[f(\theta_{\Delta t}(p)) - f(p)].$$

The vector field X^θ is called the *infinitesimal generator of* θ.

(ii) **Definition.** Let $\theta : G \to \text{Diffeo}(M)$ defined by $\theta(g) = \theta_g$ be a C^∞-action on a smooth manifold M. Then a vector field X on M is said to G-*invariant* if $(\theta_g)_*(X) = X$ for all $g \in G$.

(iii) **Theorem.** Let $\theta : \mathbb{R} \to \text{Diffeo}(M)$ defined by $\theta(t) = \theta_t$ be a C^∞-action on a smooth manifold M. Then X^θ is invariant under θ, that is, $(\theta_t)_*(X^\theta) = X^\theta$, for all $t \in \mathbb{R}$.

(iv) **Corollary.** Let $\theta : \mathbb{R} \to \text{Diffeo}(M)$ defined by $\theta(t) = \theta_t$ be a C^∞-action on a smooth manifold M. If $X_p = 0$, then for each q in the orbit of p, we have $X_q = 0$.

(v) **Theorem.** Let $\theta : \mathbb{R} \to \text{Diffeo}(M)$ defined by $\theta(t) = \theta_t$ be a C^∞-action on a smooth manifold M. The orbit of p given is either a single point or an immersion of $\mathbb{R}$ in M by the map $t \mapsto \theta_t(p)$ depending on whether or not $X_p = 0$.

(vi) **Remark.** Let $\theta : \mathbb{R} \to \text{Diffeo}(M)$ defined by $\theta(t) = \theta_t$ be a C^∞-action on a smooth manifold M. For $t_0 \in \mathbb{R}$, let $\frac{d}{dt}$ be standard basis of $T_{t_0}(\mathbb{R})$, and let $F(t) = \theta_t(p)$. Since we have

$$F_*\left(\frac{d}{dt}\right) = X_{\theta_{t_0}}(p) = X_{F(t_0)},$$

it follows that at each $p \in M$, X_p is tangent to orbit of p and is the velocity vector to $t \to F(t)$ in M.

(vii) **Definition.** Given a vector field X on a smooth manifold M, we say that a curve $F : (a,b) \to M$ is an *integral curve* of X if $\frac{dF}{dt} = X_{F(t)}$ for all $t \in (a,b)$.

(viii) **Remark.** Let $\theta : \mathbb{R} \times M \to M$ be a C^∞-action on a smooth manifold M. Then each orbit of θ is an integral curve of X^θ, that is, $\dot{\theta}(t,p) = X_{\theta(t,p)}$.

(ix) Examples of $\mathbb{R}$-actions.

(a) Let $M = \mathbb{R}^2$, and $\theta : \mathbb{R} \times M \to M$ be defined by $\theta(t,(x,y)) = (x+t, y)$. Then $X^\theta = \frac{\partial}{\partial x}$.

(b) If $M' = \mathbb{R}^2 \setminus \{(0,0)\}$, then the θ from (a) does not restrict to an action on M'. However, if we consider the open set $W \subset \mathbb{R} \times M'$ given by

$$W = \left(\bigcup_{y \neq 0} \mathbb{R} \times \{(x,y)\} \right) \cup \{(t,(x,0)) : x(x+t) > 0\},$$

then $\theta' = \theta|_W$ preserves most of the properties of θ.

(x) **Definition** Let M be a smooth manifold and $W \subset \mathbb{R} \times M$ be an open set such that for each $p \in M$, there exists real numbers $\alpha(p) < 0 < \beta(p)$ such that

$$W \cap (\mathbb{R} \times \{p\}) = (\alpha(p), \beta(p)) \times \{p\},$$

so that

$$W = \bigcup_{p \in M} (\alpha(p), \beta(p)) \times \{p\}.$$

Then a *local one-parameter action (or a flow)* on M is a C^∞ map $\theta : W \to M$ such that:

(a) $\theta_0(p) = p$ for all $p \in M$.

(b) If $(s,p) \in W$, we have:

(1) $\alpha(\theta_s(p)) = \alpha(p) - s$,

(2) $\beta(\theta_s(p)) = \beta(p) - s$, and

(3) for any $t \in (\alpha(p) - s, \beta(p) - s)$, we have $\theta_{t+s}(p) = \theta_t \circ \theta_s(p)$.

(xi) **Remark.** Let $\theta : W \to M$ be a flow on a smooth manifold M.

(a) Since W is open and $(0,p) \in W$, there exists a neighborhood $U \ni p$ such that $(-\delta, \delta) \times U \subset W$ for sufficiently small δ. Thus, θ also has a well-defined infinitesimal generator X^θ associated to it.

(b) θ satisfies $\theta_t^{-1} = \theta_{-t}$, wherever it is well-defined. In general, θ_t need not define a map on all of M.

(c) Let $V_t \subset M$ be the domain of definition of θ_t, that is, $V_t = \{p \in M : (t,p) \in W\}$. For all $p \in V_t$, we have $(\theta_t)_*(X_p^\theta) = X_{\theta_t(p)}$.

(d) The curve defined $F(t) = \theta_t(p)$, for $t \in (\alpha(p), \beta(p))$ is a C^∞ curve, which is an immersion of $(\alpha(p), \beta(p))$ if $X_p \neq 0$, and is a single point, if $X_p = 0$.

(xii) **Theorem.** Let $\theta : W \to M$ be a flow on a smooth manifold M and let $V_t \subset M$ be the domain of definition of θ_t, that is, $V_t = \{p \in M : (t, p) \in W\}$. Then:

(a) V_t is an open set for all t and

(b) $\theta_t : V_t \to V_{-t}$ is a diffeomorphism with $\theta_t^{-1} = \theta_{-t}$.

(xiii) **Theorem.** Let $\theta : W \to M$ be a flow on a smooth manifold M and let X^θ be its associated infinitesimal generator. If $p \in M$ is such that $X_p^\theta \neq 0$, then there exists a coordinate neighborhood (V, ψ) around p, a $v >$, and a corresponding neighborhood $V' \ni p$ with $V' \subset V$ such that in local coordinates $\theta|_{(-v,v)\times V'}$ is given by $(t, y_1, \ldots, y_n) \to (y_1 + t, y_2, \ldots, y_n)$. Moreover, in these coordinates, we have $X = \psi_*^{-1}\left(\frac{\partial}{\partial x_1}\right)$ for every point of V'.

2.4 Existence of integral curves

(i) **Theorem.** Suppose that for $1 \leq i \leq n$, $f_i(t, x_, \ldots, x_n)$ are C^r functions on $(-\epsilon, \epsilon) \times U$, where $r \geq 1$ and $U \subset \mathbb{R}^n$ is open. Then for each $x \in U$, there exists $\delta > 0$ and a neighborhood $V \ni x$ with $V \subset U$ such that:

(a) For each $a = (a_1, \ldots, a_n) \in V$, there exists an n-tuple of C^{r+1} functions $x(t) = (x_1(t), \ldots, x_n(t))$ with $x_i : (-\delta, \delta) \to U$ satisfying the first-order system of ODEs:

$$\frac{dx_i}{dt} = f_i(t, x),\ 1 \leq i \leq n \qquad (*)$$

with initial conditions

$$x_i(0) = a_i,\ 1 \leq i \leq n. \qquad (**)$$

(b) For each $a = (a_1, \ldots, a_n) \in V$, the $x_i(t)$ are uniquely determined in the sense that any other function $\bar{x}_i(t)$ satisfying $(*)$ and $(**)$ must agree with $x(t)$ on an open interval around $t = 0$.

(c) As the functions $x_i(t)$ are uniquely determined by $a = (a_1, \ldots, a_n) \in V$, we can write them as $x_i(t, a_1, \ldots, a_n)$ for $1 \leq i \leq n$, in which case they are of class C^r in all the variable and determine a C^r map $(-\delta, \delta) \times V \to U$.

(ii) **Definition.** If the right hand side of equation $(*)$ in Theorem 2.4(i) above is independent of t, then we say that the system of ODEs is *autonomous.*

(iii) **Remark.** Consider an autonomous system of ODEs as in Theorem 2.4(i), where the f_i depend only on $(x_1, \ldots, x_n)$.

(a) Define on $U \subset \mathbb{R}^n$ a C^∞ vector field X by $X = \sum_{i=1}^n f_i(x) \frac{\partial}{\partial x_i.}$ An integral curve of X is a smooth mapping $F : (\alpha, \beta) \to U$ such that $\dot{F}(t) = X_{F(t)}$ for all $x \in (\alpha, \beta)$. Since $F(t) = (x_1(t), \ldots, x_n(t))$, we have:

$$\dot{F}(t) = X_{F(t)} \iff \frac{dx_i}{dt} = f_i(x_1(t), \ldots, x_n(t)),\ 1 \le i \le n,$$

that is, $x(t) = (x_1(t), \ldots, x_n(t))$ are a solution of equation $(*)$.

(b) By Theorem 2.4(i), for each a in a neighborhood $V \ni x$, there exists a unique $F(t)$ satisfying $F(0) = a$ and $F : (-\delta, \delta) \to U$ for every $a \in V$.

(c) If $F(t, a) = (x_1(t, a), \ldots, x_n(t, a))$, then $\dot{x}_i(t, a) = f_i(x(t, a))$ and $x_i(0, a) = a_i$, where each is x_i is C^∞ on $((-\delta, \delta) \times V)$, an open subset of $\mathbb{R} \times U$.

(iv) **Theorem.** Let X be a C^∞ vector field on a smooth manifold M. Then:

(a) For each $p \in M$, there exists a neighborhood V and a real number $\delta > 0$ such that there exists a C^∞ mapping:

$$\theta^V : (-\delta, \delta) \times V \to M$$

satisfying

$$\dot{\theta}^V(t, q) = X_{\theta^V(t,q)}$$

and $\theta^V(0, q) = q$ for all $q \in V$.

(b) If $F(t)$ is an integral curve of X with $F(0) = q \in V$, then $F(t) = \theta^V(t, q)$, for all $|t| < \delta$. In particular, this mapping is unique in the sense that if (V_1, δ_1) is another such pair for $p \in M$, then $\theta^V = \theta^{V_1}$ on the common part of their domains.

(v) **Theorem.** Let X be a C^∞ vector field on a smooth manifold M. Then for each $p \in M$, there exists a uniquely determined open interval $(\alpha(p), \beta(p))$ having the following properties:

(a) There exists a C^∞ integral curve $F(t)$ defined on $(\alpha(p), \beta(p))$ such that $F(0) = p$.

(b) If G is another integral curve with $G(0) = p$, then the interval of definition of G is contained in $(\alpha(p), \beta(p))$ and $F(t) \equiv G(t)$ on this interval.

(vi) **Remark.** Let X be a C^∞ vector field on a smooth manifold M. By Theorem 2.4 (v), two curves of X defined on open intervals I_1 and I_2 that coincide on $I_1 \cap I_2 \neq \emptyset$, define an integral curve on $I_1 \cup I_2$. So, let $F(t) = \theta^X(t, p)$ be the unique maximal integral curve such that $F(0) = p$ and let $W = \bigcup_{p \in M} (\alpha(p), \beta(p)) \times \{p\}$. Then:

(a) W and θ^X are uniquely determined by X, and W is the domain of θ^X.

(b) W and θ^X satisfy the following properties.

(1) We have $\{0\} \times M \subset W$ and $\theta^X(0, p) = p$ for all $p \in M$.

(2) For each $p \in M$, if $\theta_p^X(t) = \theta^X(t, p)$, then $\theta_p^X : (\alpha(p), \beta(p)) \to M$ is C^∞ maximal integral curve.

(3) For each $p \in M$, there exists a neighborhood $V \ni p$ and a $\delta > 0$ such that $(-\delta, \delta) \times V \subset W$ and θ^X is C^∞ on $(-\delta, \delta) \times V$.

(vii) **Corollary.** In the notation of Remark 2.4 (vi) above, let $s \in (\alpha(p), \beta(p))$ and $q = \theta_p^X(s) = \theta^X(s, p)$ be the corresponding point of the integral curve determined by p. Then:

(a) $\alpha(q) = \alpha(p) - s$ and $\beta(q) = \beta(p) - s$. Thus, $t \in (\alpha(q), \beta(g))$ if and only if $t + s \in (\alpha(p), \beta(p))$ and

(b) $\theta^X(t, \theta^X(s, p)) = \theta^X(t + s, p)$.

(viii) **Theorem.** Let X be a C^∞ vector field on a smooth manifold M. Then:

(a) The domain W of θ^X is open in $\mathbb{R} \times M$ and

(b) θ^X is C^∞ onto M.

(ix) **Definition.** Let M be a smooth manifold, and for $i = 1, 2$, let $\theta_i : W_i \to M$ be one-parameter group actions (or flows) on M. Then we say $\theta_1 \cong \theta_2$ if $\theta_2(x) = \theta_2(x)$ for all $x \in W_1 \cap W_2$.

(x) **Theorem.** Let M be a smooth manifold.

(a) For $i = 1, 2$, let $\theta_i : W_i \to M$ be one-parameter group actions (or flows) on M. Then: $\theta_1 \cong \theta_2$ if and only if $X^{\theta_1} = X^{\theta_2}$.

(b) Furthermore, every C^∞ vector field X is the infinitesimal generator of a unique flow $\theta^X : W \to M$ (called the *maximal flow generated by* X) whose domain W is maximal among all $\tilde{\theta} \cong \theta$.

(xi) **Lemma.** Let $\theta^X : W \to M$ be the flow with maximal domain W and infinitesimal generator X acting on a smooth manifold M. For $p \in M$, let $\theta^X_p : (\alpha(p), \beta(p)) \to M$ defined by $\theta^X_p(t) = \theta^X(t,p)$ be the integral curve of X through p. If $\beta(p) < \infty$ and $\{t_n\} \subset (\alpha(p), \beta(p))$ is a sequence such that $t_n \to \beta(p)$, then $\{\theta^X(t_n, p)\}$ cannot lie on a compact set. In particular, $\{\theta^X(t_n, p)\}$ cannot approach a limit in M. A similar statement holds for $\alpha(p)$ with $\alpha(p) < \infty$.

(xii) **Corollary.** Let $\theta^X : W \to M$ be the flow with maximal domain W and infinitesimal generator X acting on a smooth manifold M. For $p \in M$, let $\theta^X_p : (\alpha(p), \beta(p)) \to M$ defined by $\theta^X_p(t) = \theta^X(t,p)$ be the integral curve of X through p.

(a) If $(\alpha(p), \beta(p))$ is a bounded interval, then the integral curve $\{\theta^X_p(t) : t \in (\alpha(p), \beta(p))\}$ is a closed subset of M.

(b) If $X_p = 0$, then $(\alpha(p), \beta(p)) = \mathbb{R}$ and if $X = 0$ outside a compact subset of M, then $W = \mathbb{R} \times M$.

(xiii) **Definition.** A C^∞ vector field X on a smooth manifold M is *complete* if it generates a global action of $\mathbb{R}$ on M, that is, the domain of θ^X is $\mathbb{R} \times M$.

(xiv) **Corollary.** If M is a compact smooth manifold, then every vector field on M is complete.

(xv) **Theorem.** Let X be a C^∞ vector field on a smooth manifold M and let $F : M \to M$ be a diffeomorphism. Then $\theta^X : W \to M$ be the maximal flow generated by X. Then X is invariant under F if and only if $F(\theta(t,p) = \theta(t, F(p))$, whenever both sides are well-defined.

(xvi) **Remark.** The main assertion in Theorem 2.4 (xv) can equivalently stated as $F_*(X) = X$ if and only if $\theta_t \circ F = F \circ \theta_t$ for all $t \in V_t$.

(xvii) **Corollary.** A left invariant vector field on a Lie group G is complete.

2.5 One-parameter subgroups

(i) **Definition.** Let G be a Lie group. A *one-parameter subgroup* of G is the image $F(\mathbb{R}$ of some Lie group homomorphism $F : \mathbb{R} \to G$.

(ii) **Remark.** Let G be Lie group and let $F : \mathbb{R} \to G$ be a Lie group homomorphism. If $\varphi : G \times M \to M$ is an action of G on M, then φ induces an $\mathbb{R}$-action $\varphi_F : \mathbb{R} \times M \to M$ on M via F defined by $\varphi_F(t, p) = \varphi(F(t), p)$.

(iii) Example of one-parameter actions.

(a) Let $G = \mathrm{GL}(3, \mathbb{R})$. Consider the homomorphism $F_1 : \mathbb{R} \to G$ defined by

$$F_1(t) = \begin{pmatrix} e^{at} & 0 & 0 \\ 0 & e^{at} & 0 \\ 0 & 0 & e^{at} \end{pmatrix},$$

and homomorphism $F_2 : \mathbb{R} \to G$ be defined by

$$F_2(t) = \begin{pmatrix} 1 & at & bt + \frac{1}{2}act^2 \\ 0 & 1 & ct \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $\mathrm{GL}(3, \mathbb{R})$ has a natural action on $\mathbb{R}^3$, by Remark 2.5 (ii), each F_i induces an action of $\mathbb{R}$ on $\mathbb{R}^3$. For example F_1 induces that action $\theta_1(t, x_1, x_2, x_3) = (e^{at}x_1, e^{at}x_2, e^{at}x_3)$ with $X_x^\theta = \dot{\theta}(a, x) = \sum_{i=1}^3 ax_i \frac{\partial}{\partial x_i}$.

(b) Consider the homomorphism $F : \mathbb{R} \to \mathrm{SO}(3)$ defined by

$$F(t) = \begin{pmatrix} \cos(at) & \sin(at) & 0 \\ -\sin(at) & \cos(at) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $\mathrm{SO}(3)$ acts on S^2 by rotations, the action induces an $\mathbb{R}$-action θ on S^2 (via F), which defines a one-parameter group of rotations about the x_3-axis given by:

$$\theta(t, x_1, x_2, x_3) = (x_1 \cos(at) + x_2 \sin(at), -x_1 \sin(at) + x_2 \cos(at), x_3).$$

The orbits under this action are the latitudes of S^2 and X^θ is tangent to them and orthogonal to the x_3-axis.

(c) A Lie group acts on itself by right translation (multiplication) defined by $\varphi : G \to \text{Diffeo}(G)$ given by $\varphi(a) = R_a$. Then φ induces an $\mathbb{R}$-action $\theta : \mathbb{R} \times G \to G$ via a homomorphism $F : \mathbb{R} \to G$ given by

$$\theta(t, g) = R_{F(t)}(g) = gF(t).$$

(iv) **Theorem.** Let $F : \mathbb{R} \to g$ be a one-parameter subgroup of a Lie group G and let X be left-invariant vector field on G defined by $X_e = \dot{F}(0)$. Then $\theta(t, g) = R_{F(t)}(g)$ defines an action $\theta : \mathbb{R} \times G \to G$ such that $X^\theta = X$. Conversely, let X be a left-invariant vector field and $\theta : \mathbb{R} \times G \to G$ be the corresponding flow generated by X. Then $F(t) = \theta(t, e)$ is a one-parameter subgroup of G such that $\theta(t, g) = R_{F(t)}(g)$.

(v) **Corollary.** Let G be a Lie group.

(a) There is a one-to-one correspondence between the elements of $T_e(G)$ and the one-parameter subgroups of G.

(b) For $Z \in T_e(G)$, let $\{F(t, Z) : t \in \mathbb{R}\}$, where $t \mapsto F(t, Z)$, be the unique corresponding one-parameter subgroup of G. Then $\mathbb{R} \times T_e(G) \to G$ is C^∞ and satisfies $F(t, sZ) = F(st, Z)$.

2.6 One-parameter subgroups of Lie groups

(i) **Definition.** The exponential e^X of a matrix $X \in M_n(\mathbb{R})$ is defined by:

$$e^X = 1 + \frac{X}{1!} + \frac{X^2}{2!} + \dots, \qquad (\dagger)$$

whenever the series converges.

(ii) **Theorem.** Consider the series (†) in Definition 2.6 (i) above.

(a) The series converges absolutely for all $X \in M_n(\mathbb{R})$ and uniformly on all compact subsets of $M_n\mathbb{R})$.

(b) The mapping $\exp : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ defined by $\exp(A) = e^{tA}$ is C^∞ and $\text{Im}\exp \subset \text{GL}(n, \mathbb{R})$.

(c) If $A, B \in M_n(\mathbb{R})$ such that $AB = BA$, then $\exp(A + B) = \exp(A)\exp(B)$.

(iii) **Corollary.** For an $A \in M_n(\mathbb{R})$, consider the map $F : \mathbb{R} \to \text{GL}(n, \mathbb{R})$ defined by $F(t) = e^{tA}$.

(a) $F(\mathbb{R})$ is an one-parameter subgroup of $\mathbb{R}$ whose corresponding vector field is given by

$$\sum_{i,j} a_{ij} \left(\frac{\partial}{\partial x_{ij}}\right)_{I_n}.$$

(b) All one parameter subgroups are of this form. Moreover, $\dot{F}(0) = A = (a_{ij})$.

(iv) **Theorem.** Let G be a Lie group and let $H < G$ be a Lie subgroup. Then the one parameter subgroups of H are those one-parameter subgroups $F(\mathbb{R}) < G$ such that $\dot{F}(0) \in T_e(H)$ considered as a subspace of $T_e(G)$.

(v) **Corollary**. Let $G = \mathbb{GL}(n, \mathbb{R})$ and let $H < G$ be a Lie subgroup.

(a) The one-parameter subgroups H are all of form $F(\mathbb{R})$, where $F(t) = e^{tA}$.

(b) Moreover the entries of $A = (a_{ij})$ are components of the vector

$$\dot{F}(0) = \sum_{i,j} a_{ij} \left(\frac{\partial}{\partial x_{ij}}\right)_e \in T_e(G),$$

which is tangent to H at e.

(vi) Examples of one-parameter subgroups.

(a) If $A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \in M_n(\mathbb{R})$, then

$$e^{tA} = \begin{pmatrix} 1 & ta & \frac{1}{2}act^2 \\ 0 & 1 & tc \\ 0 & 0 & 1 \end{pmatrix} \in \text{GŁ}(n, \mathbb{R}).$$

(b) Consider $H = \text{O}(n) < G = \text{GL}(n, \mathbb{R})$. Then

$$\mathfrak{o}(n) = \{A \in M_n(\mathbb{R}) : e^{tA} \in H, \forall t\} = \{A \in M_n(\mathbb{R}) : A^T = -A\}.$$

Hence, $\dim(\mathfrak{o}(n)) = n(n-1)/2$. A neighborhood of $O \in \mathfrak{o}(n)$ is mapped diffeomorphically by $X \mapsto e^x$ to a neighborhood of $I_n \in \text{O}(n)$.

(vii) **Definition.** The *exponential mapping* $\exp : T_e(G) \to G$ is given by $\exp(Z) = F(1, Z)$, where for $Z \in T_e(G)$, $t \mapsto F(t, Z)$ is unique one-parameter subgroup determined by Z.

(viii) **Theorem.** Let G be a Lie group.

(a) The exponential mapping $\exp : T_e(G) \to G$ is C^∞.

(b) For $Z \in T_e(G)$, let $\{F(t, Z) : t \in \mathbb{R}\}$, where $t \mapsto F(t, Z)$, be the unique one-parameter subgroup of G such that $\dot{F}(0) = Z$.

(c) The Jacobian matrix of exp at 0 is the identity matrix, that is, $\exp_*$ is the identity.

(d) If G is a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$, then for each $Z \in T_e(G)$, there exists $A = (a_{ij}) \in M_n(\mathbb{R})$ such that

$$Z = \sum_{i,j} a_{ij} \left(\frac{\partial}{\partial x_{ij}} \right)_e .$$

Moreover, for this Z, we have $\exp(tZ) = e^{tA}$.

2.7 Lie algebra of vector fields

(i) **Notation.** Let M be a smooth manifold. We denote by $\mathfrak{X}(M)$, the module over $C^\infty(M)$ of all C^∞ vector fields on M.

(ii) We say a vector space $\mathscr{L}$ over $\mathbb{R}$ is a (real) *Lie algebra* if in addition to its vector space structure, it possesses a product map $\mathscr{L} \times \mathscr{L} \to \mathscr{L}$ taking the pair (X, Y) to the elements $[X, Y]$ of $\mathscr{L}$ that satisfies the following properties.

(a) It is bilinear over $\mathbb{R}$: That is, for any $\alpha, \beta \in \mathbb{R}$ and $X_i, Y_i \in \mathscr{L}$ for $i = 1, 2$, we have:

(1) $[\alpha X_1 + \beta X_2, Y] = \alpha[X_1, Y] + \beta[X_2, Y]$.

(2) $[X, \alpha Y_1 + \beta Y_2] = \alpha[X, Y_1] + \beta[X, Y_2]$.

(b) It is skew-commutative: That is for any $X, Y \in \mathscr{L}$, we have:

$$[X, Y] = -[Y, X].$$

(c) It satisfies the Jacobi identity: That is, for any $X, Y, Z \in \mathscr{L}$, we have:

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0.$$

(iii) Examples of Lie algebras.

(a) The vector space $\mathbb{R}^3$ with the usual vector cross product $\times$ is a Lie algebra.

(b) The vector space $M_n(\mathbb{R})$ with the product defined by $[X,Y] = XY - YX$, for $X, Y \in M_n(\mathbb{R})$, is a Lie algebra.

(iv) **Remark.** Let M be a smooth manifold. In general, given $X, Y \in \mathfrak{X}(M)$, the product XY, considered as an operator on M, does not determine a C^∞ vector field.

(v) **Lemma.** Let M be a smooth manifold. Given $X, Y \in \mathfrak{X}(M)$, we have $XY - YX \in \mathfrak{X}(M)$ according to the prescription

$$(XY - YX)_p f = X_p(Yf) - Y_p(Xf),$$

where $f \in \mathscr{C}^\infty(p)$ and $Xf, Yf \in \mathscr{C}^\infty(p)$ are defined by $(Xf)(q) := X_q(f)$ and $(Yf)(q) := Y_q(f)$, for every q in some neighborhood of $U \ni p$.

(vi) **Theorem** For a smooth manifold M, the space $\mathfrak{X}(M)$ with the product $(X,Y) \mapsto [X,Y]$ is a Lie algebra.

(vii) **Definition.** Let M be a smooth manifold and let $X, Y \in \mathfrak{X}(M)$. Let $\theta^X : W \to M$ be the maximal flow generated by X. Then Lie derivative of Y with respect to X, is the vector field $L_X Y \in \mathfrak{X}(M)$ defined by:

$$(L_X Y)_p = \lim_{t\to 0} \frac{1}{t}\left[(\theta^X_{-t})_*(Y_{\theta^X(-t,p)}) - Y_p\right] = \lim_{t\to 0} \frac{1}{t}\left[Y_p - (\theta^X_t)_*(Y_{\theta^X(-t,p)})\right],$$

at each $p \in M$.

(viii) **Remark.** Let M be a smooth manifold and let $X, Y \in \mathfrak{X}(M)$.

(a) The tangent vector $(L_X Y)_p$ measures the rate of change of Y in direction of X along an integral curve of the vector field through p.

(b) If $Z_p(t) = (\theta^X_{-t})_*(Y_{\theta^X(-t,p)}) \in T_p(M)$, viewed as a curve in $\mathbb{R}^n$, then $L(XY)_p = \dot{Z}_p(0)$.

(ix) **Lemma.** Let M be a smooth manifold and let $X \in \mathfrak{X}(M)$. Let $\theta^X : W \to M$ be the maximal flow generated by X. Given $p \in M$ and $f \in C^\infty(U)$, where $U \ni p$ is an open set, we choose a $\delta > 0$ and a neighborhood $V \ni p$ such that $\theta^X((-\delta,\delta) \times V)) \subset U$. Then there exists a C^∞ function $g(t,q)$ defined on $(-\delta,\delta) \times V$ such that for $q \in V$ and $t \in (-\delta,\delta)$, we have:

$$f(\theta_t(q)) = f(q) + tg(t,q) \text{ and } X_q(f) = g(0,q).$$

(x) **Theorem.** Let M be a smooth manifold and let $X, Y \in \mathfrak{X}(M)$. Then we have:

$$L_X Y = [X,Y].$$

(xi) **Theorem.** Let N, M be smooth be smooth manifolds, and let $F : N \to M$ be a smooth mapping. For $i = 1,2$ let $X_i \in \mathfrak{X}(N)$ and $Y_i \in \mathfrak{X}(M)$ be vector fields such that $F_*(X_i) = Y_i$. Then:

$$F_*[X_1, X_2] = [F_*(X_1), F_*(X_2)].$$

(xii) **Corollary.**

(a) The left-invariant vector fields on a Lie group G form a Lie algebra $\mathfrak{g}$ with product $(X,Y) \mapsto [X,Y]$ and $\dim(\mathfrak{g}) = \dim(G)$.

(b) If $F : G_1 \to G_2$ is a homomorphism of Lie groups, then $F_* : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a homomorphism of Lie algebras.

(xiii) **Remark.** Let G be e Lie group, $H < G$ is a Lie subgroup, and $i : H \to G$ the inclusion. Then $i_*(\mathfrak{h})$ is a subalgebra of $\mathfrak{g}$, which consists of the elements of $\mathfrak{g}$ tangent to H and to its cosets gH.

(xiv) **Theorem.** Let M be a smooth manifold and let $X, Y \in \mathfrak{X}(M)$. Then $[X,Y] = 0$ if and only if for each $p \in M$, there exists $\delta_p > 0$ such that

$$\theta_s^X \circ \theta_t^Y(p) = \theta_t^X \circ \theta_s^Y(p),$$

for all $|t|, |s| < \delta_p$.

2.8 Frobenius Theorem

(i) **Definition.** Let M be a smooth manifold and let $\dim(M) = n + k$. For each $p \in M$, we assign an n-dimensional subspace $\Delta_p \subset T_p(M)$.

(a) Suppose in a neighnborhood of each $p \in M$, there exists n linearly independent C^∞ vector fields $X_1, \dots, X_n \in \mathfrak{X}(M)$, which forms basis for all $q \in U$. Then we say that Δ is a C^∞*-plane distribution of dimension n* on M and $X_1, \dots, X_n$ is a *local basis* of Δ.

(b) We say distribution Δ is *involutive* if there exists a local basis $X_1, \dots, X_n$ in a neighborhood of each point such that:

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k, \text{ for } 1 \le i, j \le n,$$

where the $c_{ij}^k \in C^\infty(M)$.

(ii) **Definition.** Let Δ be a C^∞ distribution on a smooth manifold M, and let N be a connected smooth submanifold of M. If for each $q \in N$, we have $T_q(N) \subset \Delta_q$, then we say that N is an *intergral manifold* of Δ.

(iii) **Example of a plane distributions.**

(a) If $M = \mathbb{R}^{n+k}$ and $\Delta = \langle X_i = \frac{\partial}{\partial x_i} : 1 \le i \le n \rangle$. Then the distribution is the subspace of dimension n consisting of all vectors parallel to $\mathbb{R}^n$ at each $q \in M$.

(b) Let G be e Lie group, $H < G$ is a Lie subgroup, and $i : H \to G$ the inclusion. Then the subalgebra $i_*(\mathfrak{h})$ of $\mathfrak{g}$ defines a left-invariant distribution Δ on G such that $\Delta_h = \Delta_h(H)$ for all $h \in H$.

(iv) **Definition.** Let Δ be a C^∞ distribution on a smooth manifold M and let $\dim(M) = n + k$. We say that Δ is *completely integrable* if each $p \in M$ has a cubical neighborhood (U, φ) such that $E_i = \varphi_*^{-1}\left(\frac{\partial}{\partial x_i}\right)$ for $1 \le i \le n$, are a local basis on U for Δ.

(v) **Remark.** Let Δ be a C^∞ completely integrable distribution on a smooth manifold M as in Definition 2.4 (iv). Then there exists an integral manifold N through each $q \in U$ such that $T_q(N) = \Delta_q$, that is, $\dim(N) = n$. In fact, $q = (a_1, \dots, a_n)$, then an integral manifold through q is an n-slice given by

$$N = \varphi^{-1}\{x \in \varphi(U) : x_j = a_j,\, n + 1 \le j \le m\}.$$

Furthermore, this distribution is involutive since:

$$[E_i, E_j] = \varphi_*^{-1}\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0,\ 1 \le i \text{ and } j \le n.$$

A coordinate neighborhood (U, φ) as above is called a *flat* with respect to Δ.

(vi) **Theorem (Frobenius).** A distribution Δ on a smooth manifold M is completely integrable if and only if its involutive.

(vii) **Corollary.** Let (U, φ) be a flat coordinate neighborhood relative to an involutive n-plane distribution Δ on M. Then any connected integrable manifold $C \subset U$ must lie on a single n-slice

$$S_a = \{q \in U : x_i(q) = a_i,\ n+1 \le i \le m\}.$$

(viii) **Theorem.** Let M be smooth manifold of dimension $n+k$ and let $N \subset M$ be an integral manifold of an involutive distribution Δ with $\dim(N) = \dim(\Delta)$. If $F(A) \subset N$ is a C^∞ mapping of a manifold A into M such that $F(A) \subset N$, then F is a C^∞ mapping into N.

(ix) **Definition.** A *maximal integral manifold* N of an involutive distribution Δ on a smooth manifold M is a connected integral manifold which contains every connected integral manifold that it intersects.

(x) **Remark.**

(a) If N is the maximal integral manifold of an involutive distribution Δ on a smooth manifold M, then $\dim(N) = \dim(\Delta)$.

(b) At most one maximal integral manifold that can pass through a point $p \in M$.

(xi) **Theorem.** Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. Then there exists a unique subgroup $H < G$ whose Lie algebra is $\mathfrak{h}$.

2.9 Homogeneous spaces

(i) **Definition.** A smooth manifold M is said to be homogeneous space of the Lie group G if there exists a C^∞ action of G on M.

(ii) Examples of homogeneous spaces.

  (a) Since the Lie group $O(n)$ has a transitive action on S^{n-1}, S^{n-1} is a homogeneous space of $O(n)$.

  (b) Since the Lie group $GL(n,\mathbb{R})$ has a transitive action on $\mathbb{R}^n \setminus \{0\}$, $\mathbb{R}^n \setminus \{0\}$ is a homogeneous space of $GL(n,\mathbb{R})$.

(iii) **Theorem.** Let G be a Lie group and H a closed Lie subgroup. Then there exists a unique C^∞ structure on G/H with the following properties.

  (a) The canonical projection $\pi : G \to G/H$ is C^∞.

  (b) Each $g \in G$ is in the image of a C^∞ section (V, σ) on G/H.

  (c) The natural action $\lambda : G \times G/H \to G/H$ is a C^∞ action and $\dim(G/H) = \dim(G) - \dim(H)$.

(iv) **Lemma.** If H is a connected Lie subgroup of a Lie group G, which is closed as a subset of G. then:

  (a) Each coset gH is closed.

  (b) There is a cubical neighborhood (U, φ) of any $g \in G$ such that for each coset $xH \in G/H$ either $xH \cap U = \emptyset$ or a $xH \cap U$ is a single connected slice.

(v) **Theorem.** Let G be a Lie group with a transitive action $\theta : G \times M \to M$ on a smooth manifold M.

  (a) The mapping $\tilde{F} : G \to M$ defined by $\tilde{F}(g) = \theta(g, a)$ is C^∞ and rank equal to $\dim(M)$ everywhere on G.

  (b) For $a \in M$, the stabilizer subgroup $H = \mathrm{Stab}_\theta(a) = \{g \in G : \theta_g(g) = a\}$ is a closed subgroup of G. Hence, G/H is a C^∞ manifold.

  (c) The mapping $F : G/H \to M$ defined by $F(gH) = \tilde{F}(g)$ is a diffeomorphism. Moreover, if $\lambda : G \times G/H \to G/H$ is the natural action of G on G/H, then $F \circ \lambda_g = \theta_g \circ F$, for all $g \in G$.

(vi) Example of Lie groups realized as closed stablilizer subgroups.

(a) We know that that $\mathrm{Isom}(\mathbb{R}^n) \cong \mathrm{O}(n) \times \mathbb{R}^n$. Consider the Lie subgroup of G of $\mathrm{GL}(n+1,\mathbb{R})$ defined by

$$G = \left\{ \begin{pmatrix} A & V^T \\ 0\ldots 0 & 1 \end{pmatrix} : A \in \mathrm{O}(n) \text{ and } V \in \mathbb{R}^n \right\}$$

and the set

$$X = \begin{pmatrix} X^T \\ 1 \end{pmatrix} : X \in \mathbb{R}^n\}.$$

Then G acts transitively on X and $\mathrm{Stab}_\theta(0) = \mathrm{O}_n$. Hence, $\mathrm{O}(n)$ is a closed subgroup of G.

(b) Consider the transitive action of the Lie group $G = \mathrm{SL}(n,\mathbb{R})$ on $\mathbb{R}P^n$ via the action $(g,[x]) \overset{\theta}{\longmapsto} [gx]$. Then:

$\mathrm{Stab}_\theta([(1,0,\ldots,0)]) = \{A = (a_{ij} \in \mathrm{SL}(n,\mathbb{R}) : a_{11} \neq 0 \text{ and } a_{i1} = 0, \text{ for } i > 1\}$.

(c) Consider the transitive action $\theta : G \times M \to M$ of the Lie group $G = \mathrm{GL}(n,\mathbb{R})$ on the Grassmanian $M = G(k,n)$, the set of k-frames through the origin. For a k-plane $P \in M$, let $H = \mathrm{Stab}_\theta(P)$. Then $G/H \cong G(k,n)$ and hence $G(k,n)$ is a manifold.

(vii) **Remark.** If a Lie group acts transitively on set X in such a way that the stabilizer subgroup of a point $a \in X$ is a closed Lie subgroup, then there exists a unique C^∞ structure on X such that the action is C^∞.

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